KIAS summer course: Tropical Geometry

Exercises for discussion on Thursday, Friday and beyond

1. Consider a general arrangement of n tropical hyperplanes in $\mathbb{TP}^d$. How many components are there in the complement of this arrangement ?

2. Show that every balanced one-dimensional fan in $\mathbb{R}^3$ is the tropicalization of a space curve in $\mathbb{C}^3$. More precisely, let $v_1, v_2, \ldots, v_n$ be integer vectors in $\mathbb{Z}^3$ such that $v_1 + v_2 + \cdots + v_n = 0$, and let Σ be the union of the n rays $\mathbb{R}_{\geq 0} v_i$. Construct a prime ideal $I \subset \mathbb{C}[x, y, z]$ with $\mathcal{T}(I) = \Sigma$.

3. Consider the hypersurface in $\mathbb{P}^4$ given by the parametrization

   $a = u^4 + x^4$, $b = u^3 v + x^3 y$, $c = u^2 v^2 + x^2 y^2$, $d = uv^3 + xy^3$, $e = v^4 + y^4$.

   Determine the implicit equation of this hypersurface, compute the corresponding tropical hypersurface, and draw this hypersurface as a graph. Next consider the image in $\mathbb{TP}^4$ of the tropical parametrization

   $$A = U^3 \oplus X^4, B = U^3 V \oplus X^3 Y, C = U^2 V^2 \oplus X^2 Y^2$$
   $$D = UV^3 \oplus XY^3, E = V^4 \oplus Y^4.$$

   Compute this image, and draw it as a subgraph of the previous graph.

4. Let f and g be Laurent polynomials in two variables with generic coefficients, with Newton polygons P and Q respectively. Use tropical geometry to give a proof of Bernstein's Theorem: *The number of solutions of $f = g = 0$ in $(\mathbb{C}^*)^2$ equals* $\text{area}(P) + \text{area}(Q) - \text{area}(P + Q)$. What about the (stable) intersection of n tropical hyperplanes in $\mathbb{R}^n$?

5. In your opinion, is the following statement true or false: *The 4×4-minors of a 5×5-matrix are a tropical basis for the ideal they generate.*

6. The following 3×6-matrix has tropical rank three:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 6 & 8 & 10 \end{pmatrix}.$$

Its tropical column span defines a tropical hexagon in the plane $\mathbb{TP}^2$, while its tropical row span defines a tropical triangle in $\mathbb{TP}^5$. Compute these two objects, draw them, and show that they are isomorphic.

7. Consider the family of plane cubic curves of the special form

$$c_1x^2y + c_2xy^2 + c_3x^2 + c_4xy + c_5y^2 + c_6x + c_7y \quad = \quad 0.$$

(a) Compute the Newton polytope of the discriminant of this family.
(b) How many rational curves of the above special form pass through five general points in the complex projective plane?
(c) Draw a picture that demonstrates the tropical solution to this curve counting problem, i.e., pick five general points in the plane $\mathbb{TP}^2$ and determine all tropical curves of genus zero passing through your points.