Tropical Mathematics

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Tropical Arithmetic

Addition and Multiplication:

$$x \oplus y \quad = \quad \text{minimum of } x \text{ and } y$$

$$x \odot y \quad = \quad x + y$$

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Neutral Elements:

$$\infty \oplus x \quad = \quad x$$
$$0 \odot x \quad = \quad x$$

Tropical Semiring $(\mathbb{R} \cup \{\infty\}, \odot, \oplus)$

Matrix Multiplication

$$\begin{bmatrix} 3 & 3 \\ 0 & 7 \end{bmatrix} \odot \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix} =$$

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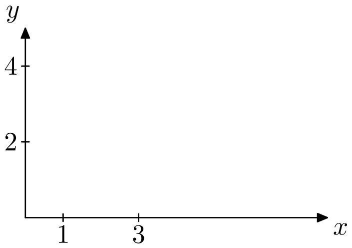

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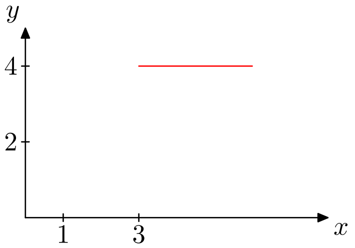

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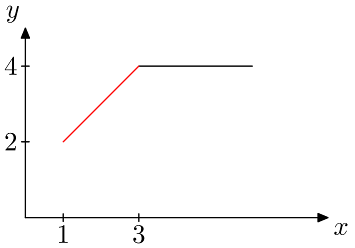

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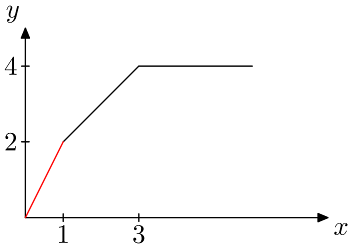

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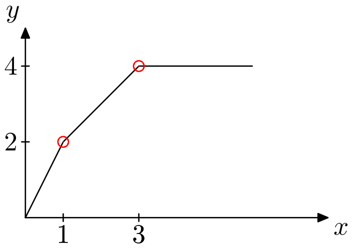

Quadratic Formula

$$x^2 \;\oplus\; a \odot x \;\oplus\; b \;=$$

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$$x^2 \ \oplus \ a \odot x \ \oplus \ b \ = \ \begin{cases} (x \oplus a) \odot (x \oplus (b - a)) & \text{if } 2a \leq b, \\ \quad (x \oplus \frac{b}{2})^2 & \text{otherwise.} \end{cases}$$

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Note: Two different polynomials can represent the same *function*.

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Fundamental Theorem of Algebra

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Fundamental Theorem of Algebra

Every tropical polynomial function $f(x)$ of degree n is uniquely the product of n linear polynomials $x \oplus c_i$ times a constant.

Solving Cubic Equations

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Q2: Let $K = \overline{\mathbb{Q}(\varepsilon)}$.
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A2:

$$\begin{aligned} x \;=\; & \varepsilon^2 - \varepsilon^4 - \varepsilon^6 - \varepsilon^7 - 2\varepsilon^8 + \ldots \\ & \varepsilon^4 - \varepsilon^5 - 3\varepsilon^7 - 3\varepsilon^8 - 16\varepsilon^9 + \ldots \\ & \varepsilon^5 + \varepsilon^6 + 2\varepsilon^7 + 5\varepsilon^8 + 13\varepsilon^9 + \ldots \end{aligned}$$

Plane Geometry

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Given a tropical polynomial $f(x, y)$, its curve $\mathcal{T}(f)$ is the set of points $(x, y) \in \mathbb{R}^2$ where the minimum is attained *twice*.

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Lines:
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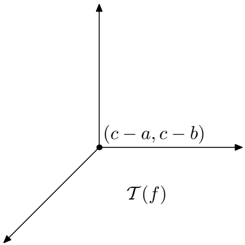

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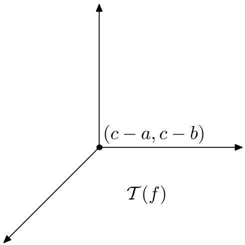

Fact 1: Any two points span a unique line.

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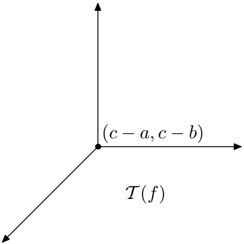

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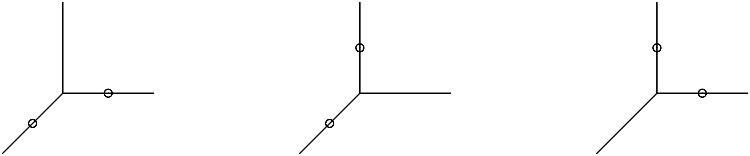

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Fact 2: Any two lines meet in a unique point.

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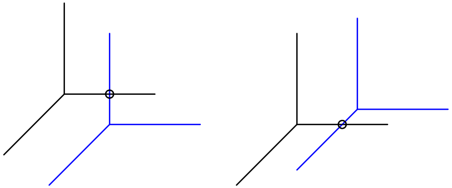

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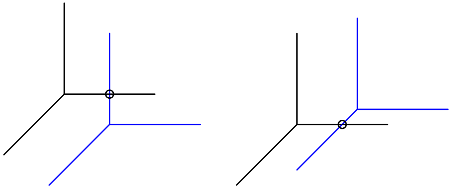

Q: Does Pappus' Theorem hold tropically?

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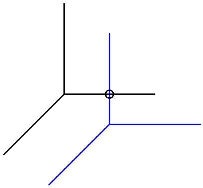

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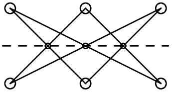

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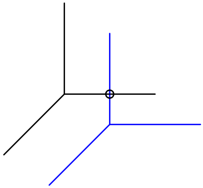

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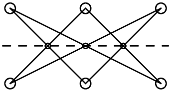

A: No (math.AG/0306366)
Yes (math.AG/0409126)

Quadratic Curves

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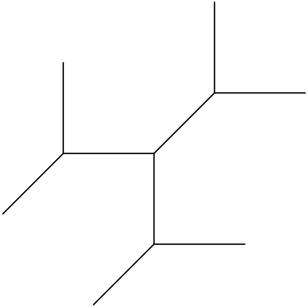

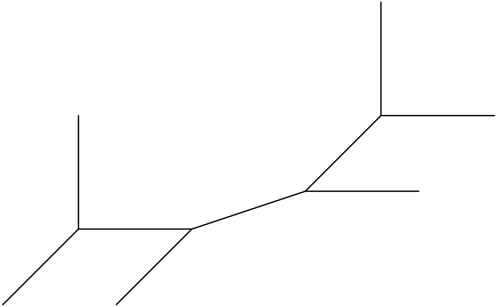

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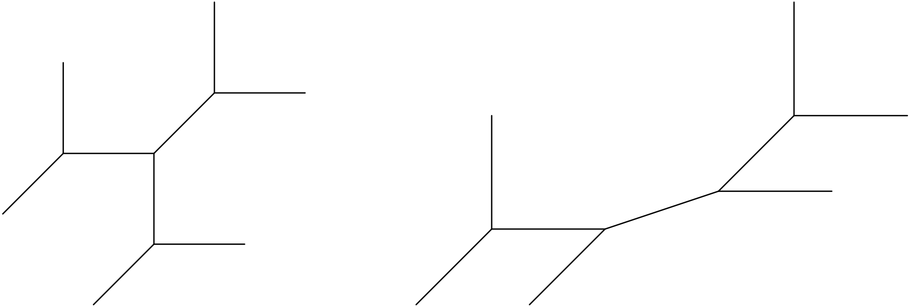

Balanced graph with two parallel halfrays in each direction.

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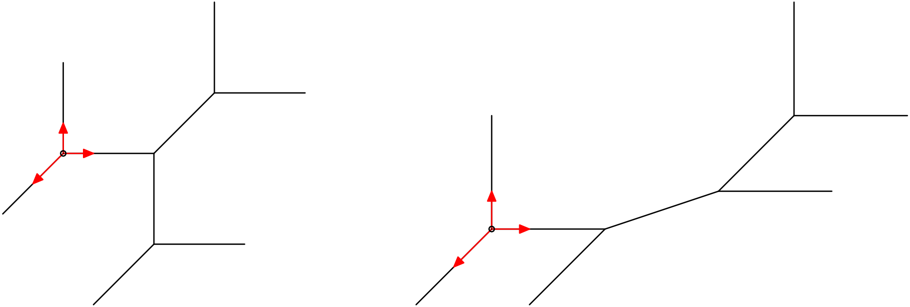

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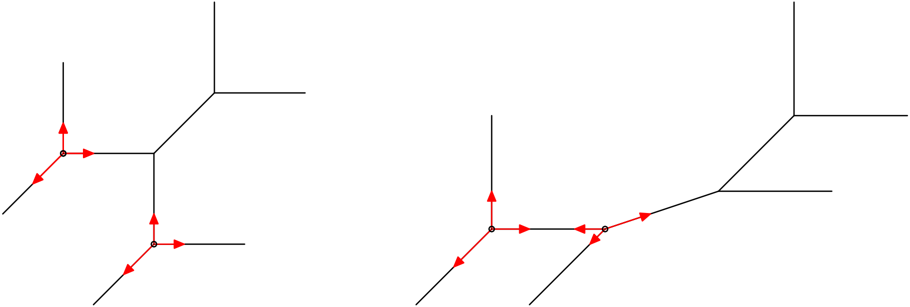

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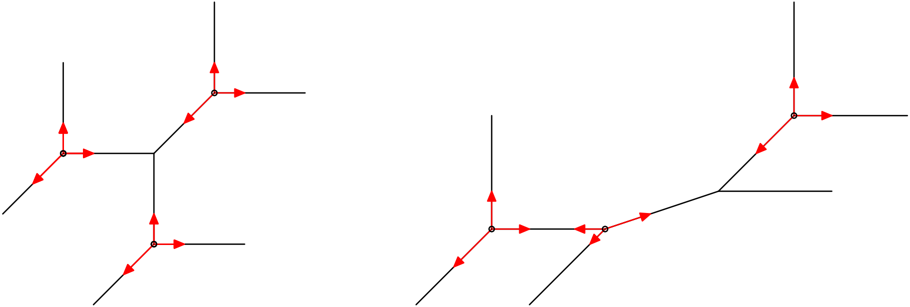

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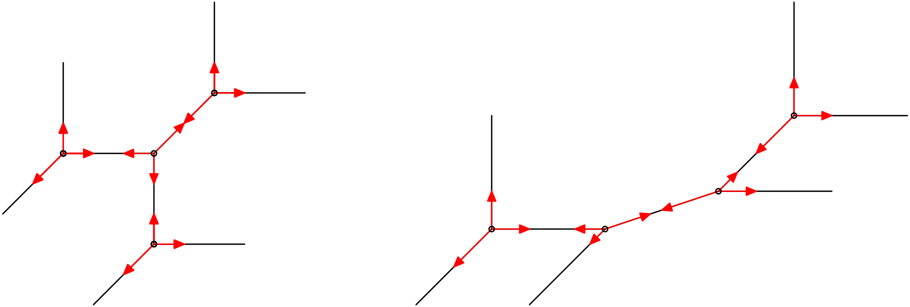

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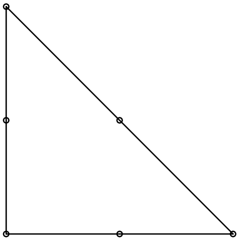
Quadratic Curves

Dual to subdivided Newton triangle.



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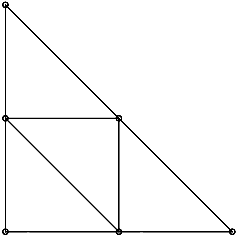

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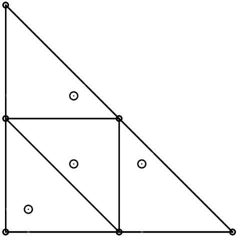

- One vertex for each bounded region.

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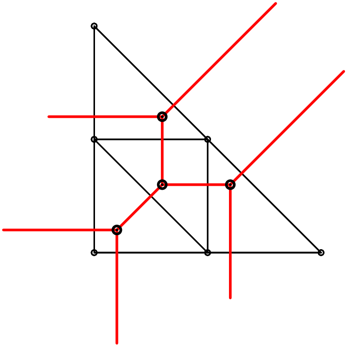

- One vertex for each bounded region.
- One edge connecting each pair of adjacent regions.

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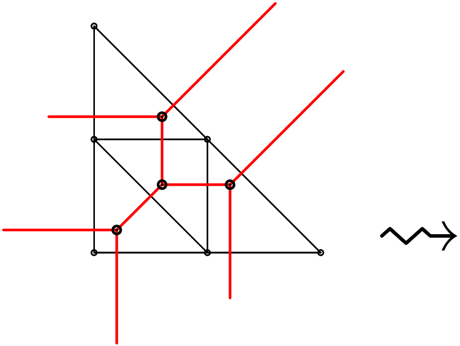

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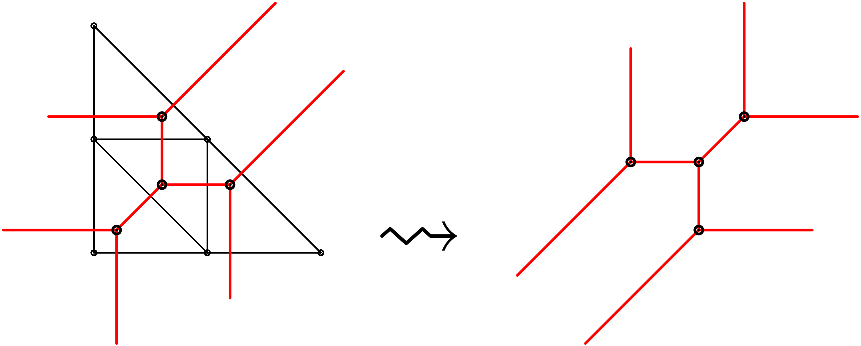

- One vertex for each bounded region.
- One edge connecting each pair of adjacent regions.
- Rotate 180°.

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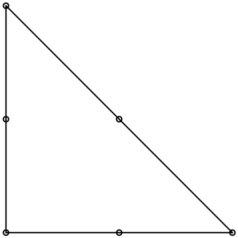

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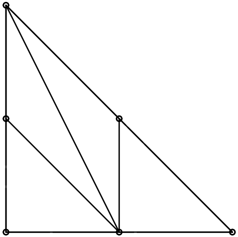

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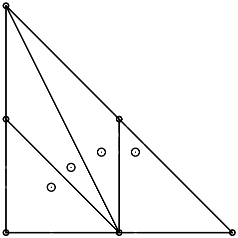

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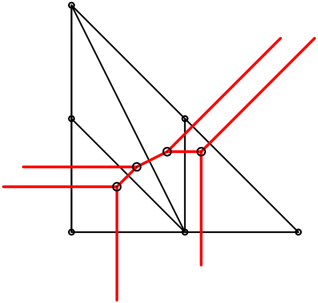

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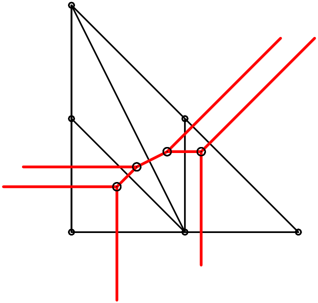

$\rightsquigarrow$

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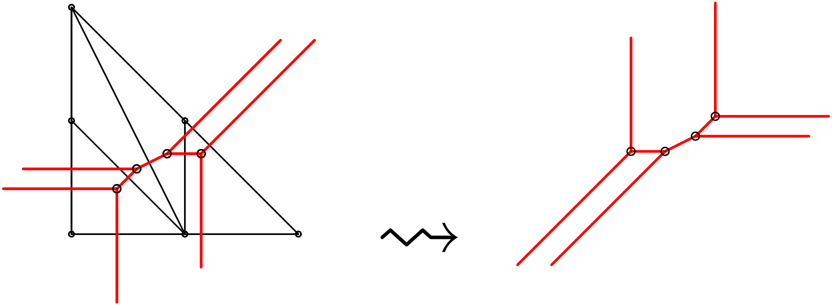

Three Facts About Plane Curves

- Through any five points in $\mathbb{R}^2$, there is a unique quadratic curve.

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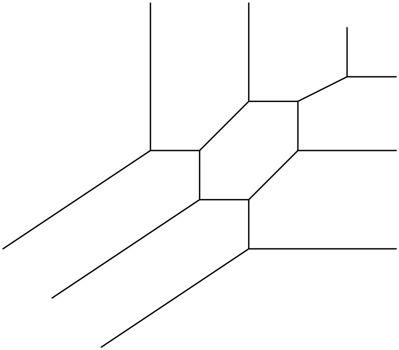
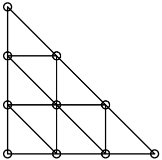
- Through any five points in $\mathbb{R}^2$, there is a unique quadratic curve.
- Cubic curves are elliptic curves (genus 1)





Three Facts About Plane Curves

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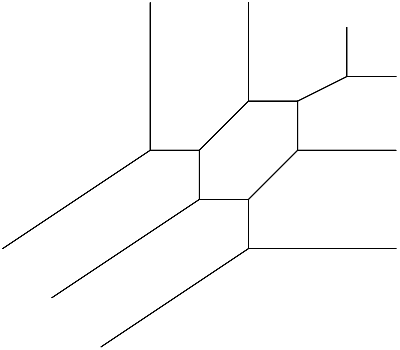
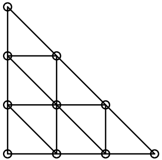
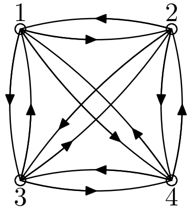




- **Bézout's Theorem**: Two plane curves of degree d and e always intersect in $d \cdot e$ points.

Matrices and Metrics

$$D = \begin{bmatrix} 0 & d_{12} & d_{13} & d_{14} \\ d_{21} & 0 & d_{23} & d_{24} \\ d_{31} & d_{32} & 0 & d_{34} \\ d_{41} & d_{42} & d_{43} & 0 \end{bmatrix}$$



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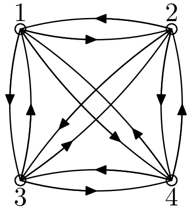

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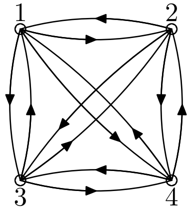

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- To find shortest pairwise distances in a directed graph D with n nodes, compute the tropical matrix power D^n.

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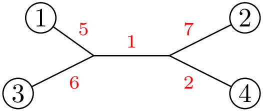

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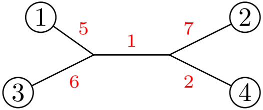

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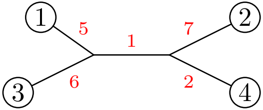

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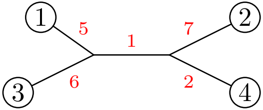

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A: No, but biologists care about those that are.

Phylogenetics

Theorem [4 Point Condition]:
A metric D is a tree metric if and only if

$$-D \in \mathcal{T}(d_{ij} \odot d_{kl} \oplus d_{ik} \odot d_{jl} \oplus d_{il} \odot d_{jk})$$

for any four taxa i, j, k and l.

Proof: [ASCB, Theorem 2.34]

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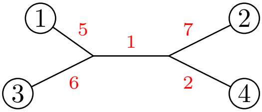

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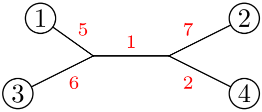

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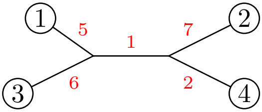

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Theorem: The space of trees equals the tropical Grassmannian $\mathbb{G}(2, n)$.

What Next?

Review what you have seen in this lecture:

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Research Program on Tropical Geometry

August 17 to December 18, 2009

Co-organizers: Eva-Maria Feichtner, Ilia Itenberg, and Grigory Mikhalkin.