KIAS summer course: Tropical Geometry

Exercises for discussion on Tuesday and Wednesday

1. Can you find 2 tetrahedra in $\mathbb{R}^3$ whose Minkowski sum has 16 vertices? Can you find 3 triangles in $\mathbb{R}^3$ whose Minkowski sum has 27 vertices?

2. Let Σ be the 2-skeleton of the boundary of the 5-dimensional cube. Determine the integral homology of Σ. How about the homotopy type?

3. Let $K = \overline{\mathbb{Q}(t)}$. The following ideal in $K[x, y]$ describes the intersection of a "moving circle" and a "moving hyperbola" in the plane:
$$I \quad = \quad \big\langle\, (x - 3t)^2 + (y - 7/t)^2 - t\,,\; xy - t^3 \,\big\rangle.$$
Compute the variety $V(I)$. Represent each point by Puiseux series.

4. Download the `Singular` package `tropical.lib` due to Thomas Markwig, and illustrate its features by computing some non-trivial examples.

5. Determine the polyhedral subdivision $\Sigma(F)$ of $\mathbb{R}^3$ given by the tropical polynomial $F(w_1, w_2, w_3) \;=\; (w_1 \oplus w_2) \odot (w_1 \oplus w_3) \odot (w_2 \oplus w_3) \;\oplus\; 1$.

6. Consider the row space of $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$ as a subvariety of $\mathbb{Q}^5$, with 2-adic valuation, and let I be its vanishing ideal in $\mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$. Compute an explicit Gröbner complex for I, and find the initial ideals $\mathrm{in}_w(I)$ associated to all vertices w of your polyhedral complex in $\mathbb{TP}^4$.

7. Does there exist a tropical cubic surface in $\mathbb{TP}^3$ that contains 27 distinct tropical lines ? What would be the f-vector of such a cubic surface ?

8. The Grassmannian $G_{2,6}$ is the intersection of the fifteen hypersurfaces $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} \;=\; 0$ for $1 \leq i < j < k < l \leq 6$. Determine the tropical variety defined by these equations, and compute its f-vector. Show that the vector $w = e_{12} + e_{34} + e_{56}$ lies in this tropical variety, and determine a preimage of w under the valuation map if $K = \mathbb{Q}(t)$.